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Normality of the maximum principle for nonconvex constrained Bolza problems [☆]

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Abstract

We consider a Bolza optimal control problem with state constraints. It is well known that under some technical assumptions every strong local minimizer of this problem satisfies first order necessary optimality conditions in the form of a constrained maximum principle. In general, the maximum principle may be abnormal or even degenerate and so does not provide a sufficient information about optimal controls. In the recent literature some sufficient conditions were proposed to guarantee that at least one maximum principle is nondegenerate, cf. [A.V. Arutyunov, S.M. Aseev, Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints, *SIAM J. Control Optim.* 35 (1997) 930–952; F. Rampazzo, R.B. Vinter, A theorem on existence of neighbouring trajectories satisfying a state constraint, with applications to optimal control, *IMA* 16 (4) (1999) 335–351; F. Rampazzo, R.B. Vinter, Degenerate optimal control problems with state constraints, *SIAM J. Control Optim.* 39 (4) (2000) 989–1007]. Our aim is to show that actually conditions of a similar nature guarantee normality of *every nondegenerate maximum principle*. In particular we allow the initial condition to be fixed and the state constraints to be nonsmooth. To prove normality we use J. Yorke type linearization of control systems and show the existence of a solution to a linearized control system satisfying new state constraints defined, in turn, by linearization of the original set of constraints along an extremal trajectory.

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1. Introduction

In the recent years many authors got interested by the regularity of optimal solutions to calculus of variations and optimal control problems. For instance their Lipschitz continuity is important for the nonoccurrence of the Lavrentiev phenomenon, which in turn allows numerical construction of optimal solutions.

For the calculus of variations problems sufficient conditions for Lipschitz continuity of optimal trajectories were derived for instance in [7,8,12] for nonconvex with respect to the velocity variable Lagrangians that may be also discontinuous with respect to the state variable. This extended some earlier results from [1], where the discontinuous Lagrangian was assumed to be convex in the velocity. In particular Cellina has shown in [7] that the Tonelli's growth condition that guarantees the Lipschitz continuity of minimizers in the calculus of variations may be replaced by a condition on the growth of the Hamiltonian. In [6] this condition was extended and generalized to the unconstrained Bolza problems in optimal control. Less is known for control problems under state constraints. Some results on Lipschitz regularity of minimizers for problems with end point constraints can be found in [11,15,21,22], where much more regularity of data is assumed. The question whether a similar result holds true in the presence of state constraints is still open, even if in [15] Lipschitz continuity was derived using normality of the maximum principle and Tonelli's growth condition.

On the other hand, it was shown in [5,14] that for a constrained Bolza optimal control problem with smooth Hamiltonian enjoying some monotonicity properties, any Lipschitz optimal trajectory satisfying the normal necessary optimality conditions is in fact much more regular (of class $C^{1,\alpha}$), where α denotes the Hölder exponent of the derivative. So it is sufficient to know both the Lipschitz continuity of an optimal trajectory and the normality of the maximum principle to deduce further smoothness of an optimal solution. The main objective of this paper is to discuss normality of the constrained maximum principle for Lipschitz optimal trajectories.

More precisely we consider the following control system under state constraints

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \text{ a.e. in } [0, 1], \\ x(t) \in K & \text{for all } t \in [0, 1], \\ (x(0), x(1)) \in C, \end{cases} \quad (1.1)$$

where $U : [0, 1] \rightsquigarrow \mathcal{Z}$ is a measurable set-valued map with nonempty closed values, $f : [0, 1] \times \mathbb{R}^n \times \mathcal{Z} \rightarrow \mathbb{R}^n$, \mathcal{Z} is a complete separable metric space, K and C are closed nonempty subsets of \mathbb{R}^n and $\mathbb{R}^n \times \mathbb{R}^n$, respectively.

If a trajectory/control pair (x, u) (with $x(\cdot)$ absolutely continuous and $u(\cdot)$ measurable) satisfies system (1.1), then it is called *admissible* (for system (1.1)).

Given $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $\ell : [0, 1] \times \mathbb{R}^n \times \mathcal{Z} \rightarrow \mathbb{R}$, we consider the constrained Bolza optimal control problem where we wish to minimize the following functional

$$J(x, u) := \varphi(x(0), x(1)) + \int_0^1 \ell(t, x(t), u(t)) dt \quad (1.2)$$

over all admissible pairs (x, u) .

If (z, \bar{u}) is a minimizer, then under some regularity assumptions, there exist $\lambda \in \{0, 1\}$, a normalized function with bounded total variation $\psi \in NBV([0, 1], \mathbb{R}^n)$ and an absolutely continuous function $p(\cdot) : [0, 1] \rightarrow \mathbb{R}^n$ not vanishing simultaneously such that $p(\cdot)$ is a solution to the adjoint system

$$-p'(s) = \frac{\partial f}{\partial x}(s, z(s), \bar{u}(s))^* (p(s) + \psi(s)) - \lambda \frac{\partial \ell}{\partial x}(s, z(s), \bar{u}(s)),$$

satisfying the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda \nabla \varphi(z(0), z(1)) + N_C(z(0), z(1)),$$

the maximum principle

$$\langle p(s) + \psi(s), z'(s) \rangle - \lambda \ell(s, z(s), \bar{u}(s)) = \max_{u \in U(s)} \langle p(s) + \psi(s), f(s, z(s), u) \rangle - \lambda \ell(s, z(s), u)$$

a.e. in $[0, 1]$ and

$$\psi(t) = \int_{[0, t]} v(s) d\mu(s) \quad \forall t \in (0, 1],$$

for a positive (scalar) Radon measure μ on $[0, 1]$ and a Borel measurable $v(\cdot) : [0, 1] \rightarrow \mathbb{R}^n$ satisfying $v(s) \in N_K(z(s)) \cap B$ μ -a.e. The constrained maximum principle is called normal if $\lambda = 1$. We call it nondegenerate if $\lambda + \sup_{t \in (0, 1)} |p(t) + \psi(t)| \neq 0$.

In general the first order necessary conditions for optimization and optimal control problems lead to multiplier rules that may have the Lagrange multiplier in front of the cost elements to be equal to zero. This corresponds in our case to $\lambda = 0$. These are the so-called abnormal necessary conditions that are not simple to be handled and usually do not allow to investigate higher order optimality conditions. For this reason it is important to provide sufficient conditions for normality. For instance in mathematical programming the Mangasarian–Fromowitz constraint qualification conditions do guarantee normality. In optimal control without state constraints, i.e. when $K = \mathbb{R}^n$, there are the so-called calmness conditions that may be found for instance in [23]. When the state constraints are present, then many authors investigated the so-called nondegeneracy conditions, see for instance [2, 13], but very little is known about normality.

In [19] Rampazzo and Vinter investigated the question of normality when the boundary of state constraints is smooth, the set of controls does not depend on time and for a continuous mapping $v : [0, 1] \times \mathbb{R}^n \rightarrow U$ the velocity $f(t, x, v(t, x))$ points strictly inside the state constraints. Under these and some other technical assumptions, they proved that every optimal trajectory–control pair satisfies a normal maximum principle, see also [16]. This applies also to problems with a fixed initial state. The aim of this paper is to propose conditions that guarantee normality of *every nondegenerate constrained maximum principle*. Some results in this direction were previously obtained in [9, 14, 15] for state constraints given as a finite intersection of smooth sets and also for convex constraints. Furthermore, it was assumed that $C = C_0 \times \mathbb{R}^n$ with $C_0 \subset \mathbb{R}^n$ having a nonempty interior. In particular, the case when C_0 is a singleton was excluded. In 2005 the second author was asked by A. Ioffe whether it is possible to get sufficient conditions for normality of *every constrained maximum principle* when the initial state is fixed and belongs to the interior of state constraints. Here we provide a positive answer to this question.

In the present paper we consider much more general constraints and we propose a simpler proof of normality of the maximum principle for every Lipschitz continuous optimal trajectory.

In order to show that a given extremal trajectory–control pair (z, \bar{u}) is normal we consider the linearized in the sense of Yorke [24] control system

$$\begin{cases} w'(t) = \frac{\partial f}{\partial x}(t, z(t), \bar{u}(t))w(t) + v(t), & v(t) \in T_{\overline{\text{co}}(f(t, z(t), U(t)))}(z'(t)) \text{ a.e. in } [0, 1], \\ w(0) = 0 \end{cases}$$

and show that it has an absolutely continuous solution $w(\cdot)$ satisfying the state constraints $w(t) \in \text{Int } T_K(z(t))$ for all $t \in (0, 1]$, where $T_K(z(t))$ denotes the tangent cone to K at $z(t)$. This is the so-called viability problem whose specific difficulty is the absence of closedness of constraints. Indeed, whenever the interior of K is nonempty the graph of the set-valued map $T_K(\cdot)$ is not closed. Once we know that a solution as above does exist, a simple analogue of Lemma 6.1 from [14] yields normality. In particular, our result applies when f is just Lipschitz with respect to the state variable and $f'_x(t, z(t), \bar{u}(t))$ is replaced by any integrable selection $A(\cdot)$ from the generalized Jacobian $\partial_x f(t, z(t), \bar{u}(t))$, i.e. $A(t) \in \partial_x f(t, z(t), \bar{u}(t))$ a.e.

2. Preliminaries

The closed ball of radius r centered at $x_0 \in \mathbb{R}^n$ is denoted by $B(x_0; r) = \{x \in \mathbb{R}^n : |x_0 - x| \leq r\}$, while for the closed unit ball in \mathbb{R}^n we simply write B . $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere in \mathbb{R}^n . For a subset $Y \subset \mathbb{R}^n$, $\text{co } Y$ (respectively $\overline{\text{co}} Y$) denotes the convex hull of Y (respectively closed convex hull of Y) and ∂Y the boundary of Y , while Y^- states for the negative polar cone to Y , that is $Y^- := \{\xi \in \mathbb{R}^n : \langle \xi, y \rangle \leq 0 \ \forall y \in Y\}$, where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the canonical scalar product in \mathbb{R}^n .

By $\text{dist}(x; K) := \inf_{y \in K} |x - y|$ we denote the distance function from a point $x \in \mathbb{R}^n$ to a subset $K \subset \mathbb{R}^n$ and by $d_K(x)$ the oriented distance given by $d_K(x) = \text{dist}(x; K)$ for $x \notin K$ and $d_K(x) = -\text{dist}(x; \mathbb{R}^n \setminus K)$ for $x \in K$. Finally, if K is a closed subset of \mathbb{R}^n , then by $T_K(x)$, $C_K(x)$ and $N_K(x)$ we denote respectively the contingent cone, the Clarke tangent cone and the (Clarke) normal cone to K at x . We recall that

$$T_K(x) = \left\{ v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x + hv; K)}{h} = 0 \right\},$$

$$C_K(x) = \left\{ v \mid \lim_{h \rightarrow 0^+, K \ni y \rightarrow x} \frac{\text{dist}(y + hv; K)}{h} = 0 \right\}$$

while $N_K(x) = C_K(x)^-$.

A closed set $K \subset \mathbb{R}^n$ is called *sleek* if the set-valued map $K \ni x \rightsquigarrow T_K(x)$ is lower semicontinuous or, equivalently, if $T_K(x) = C_K(x)$ for all $x \in K$. For a discussion on tangent and normal cones we refer the reader for instance to [4] or [10]. Here we just recall that any closed convex subset of a finite dimensional space is sleek. Furthermore, if the oriented distance d_K of a closed subset $K \subset \mathbb{R}^n$ is continuously differentiable on a neighborhood of ∂K , then K is sleek.

For any $A \in \mathbb{R}^{n \times m}$ (the space of real $(n \times m)$ -matrices), A^* denotes its adjoint and $\|A\| := \sup_{x \in S^{m-1}} |Ax|$ its norm; for a measurable essentially bounded $A : [0, 1] \rightarrow \mathbb{R}^{n \times m}$, $\|A\|_\infty := \text{ess-sup}_{s \in [0, 1]} \|A(s)\|$.

The space of absolutely continuous functions on $[0, 1]$ with values in \mathbb{R}^n is denoted by $W^{1,1} := W^{1,1}([0, 1]; \mathbb{R}^n)$, while $NBV := NBV([0, 1], \mathbb{R}^n)$ denotes the space of normalized functions of bounded variation on $[0, 1]$ with values on \mathbb{R}^n , i.e. the space of functions with bounded total variation, vanishing at zero and right-continuous on $(0, 1)$. For any $\psi \in NBV$, the right (left) limit of ψ at $t \in [0, 1)$ (respectively $t \in (0, 1]$) is denoted by $\psi(t+)$ (respectively $\psi(t-)$). For properties of the space $NBV([0, 1], \mathbb{R}^n)$ see, for instance, [17].

Given a locally Lipschitz function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, the subdifferential of φ at x is defined by

$$\partial^- \varphi(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}$$

and the limiting subgradient $\partial \varphi(x)$ of φ at x is the set of all vectors $p \in \mathbb{R}^n$ such that for some sequence x_i converging to x there exist $p_i \in \partial^- \varphi(x_i)$ satisfying $\lim_{i \rightarrow \infty} p_i = p$, see for instance [23]. We recall that if φ is convex, then $\partial \varphi(x)$ coincides with the subdifferential of convex analysis of φ at x and if $\varphi \in C^1$, then $\partial \varphi(x)$ coincides with the gradient of φ at x .

Define the Hamiltonian $H : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \times \{0, 1\} \rightarrow \mathbb{R} \cup \{+\infty\}$ associated to the above Bolza problem as follows:

$$H(t, x, p, \lambda) = \sup_{u \in U(t)} (\langle p, f(t, x, u) \rangle - \lambda \ell(t, x, u)). \quad (2.1)$$

Definition 2.1. In this paper an admissible trajectory/control pair (z, \bar{u}) of system (1.1) is called an extremal if it satisfies the constrained maximum principle for problem (1.2) in the following form: there exist $\lambda \in \{0, 1\}$, $\psi \in NBV$ and $p \in W^{1,1}$ not vanishing simultaneously such that for some integrable mappings $A : [0, 1] \rightarrow \mathbb{R}^{n \times n}$ and $g : [0, 1] \rightarrow \mathbb{R}^n$, $p(\cdot)$ is a solution to the adjoint system

$$-p'(s) = A^*(s)(p(s) + \psi(s)) - \lambda g(s) \quad \text{a.e. in } [0, 1], \quad (2.2)$$

satisfying the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda \partial \varphi(z(0), z(1)) + N_C(z(0), z(1)), \quad (2.3)$$

the maximum principle

$$\langle p(s) + \psi(s), z'(s) \rangle - \lambda \ell(s, z(s), \bar{u}(s)) = H(s, z(s), p(s) + \psi(s), \lambda) \quad \text{a.e. in } [0, 1] \quad (2.4)$$

and the jump conditions

$$\begin{aligned} \psi(0+) &\in N_K(z(0)), & \psi(t) - \psi(t-) &\in N_K(z(t)), \\ \psi(t) &= \int_{[0,t]} v(s) d\mu(s) \quad \forall t \in (0, 1], \end{aligned} \quad (2.5)$$

for a positive (scalar) Radon measure μ on $[0, 1]$ and a Borel measurable $v(\cdot) : [0, 1] \rightarrow \mathbb{R}^n$ satisfying

$$v(s) \in N_K(z(s)) \cap B \quad \mu\text{-a.e.} \quad (2.6)$$

A constrained maximum principle is normal if $\lambda = 1$. In such a case (z, \bar{u}) is called normal extremal.

A constrained maximum principle is nondegenerate if

$$\lambda + \sup_{t \in (0,1)} |p(t) + \psi(t)| \neq 0.$$

For the derivation of the maximum principle under state constraints we refer, for instance, to [23] and for the jump conditions to [14]. An equivalent notion of nondegeneracy was introduced in [2].

Remark 2.2. Equation (2.2) results from the adjoint system of some versions of the Pontryagin Maximum Principle. For instance in [23] (A, g) is a measurable selection from the generalized jacobian of $(f, \ell)(t, \cdot, \bar{u}(t))$ at $z(t)$.

3. Normality of the maximum principle

We impose first some assumptions on our constrained control system:

$$\left\{ \begin{array}{l} \text{(i) } U : [0, 1] \rightsquigarrow \mathcal{Z} \text{ is measurable with nonempty closed values,} \\ \text{(ii) } C = C_0 \times \mathbb{R}^n, \text{ where } C_0 \subset \mathbb{R}^n \text{ is closed,} \\ \text{(iii) } f \text{ and } \ell \text{ are measurable in } t, \text{ continuous in } x \text{ and } u, \\ \text{(iv) for all } R > 0 \text{ there exists } k_R \in L^1(0, 1) \text{ such that} \\ \quad f(t, \cdot, u) \text{ and } \ell(t, \cdot, u) \text{ are } k_R(t)\text{-Lipschitz on } B(0; R), \\ \text{(v) } \varphi \text{ is locally Lipschitz,} \\ \text{(vi) } \forall R > 0, \exists \eta_R > 0, M_R > 0, \rho_R > 0 \text{ such that} \\ \quad \forall (t, x) \in [0, 1] \times (\partial K \cap RB), \forall y \in K \cap B(x; \eta_R), \\ \quad \forall s \in [t - \eta_R, t + \eta_R] \cap [0, 1], \exists u_y \in U(s) \text{ satisfying} \\ \quad |f(s, y, u_y)| \leq M_R, \sup_{n \in N_K(x) \cap S^{n-1}} \langle n, f(s, y, u_y) \rangle \leq -\rho_R, \end{array} \right. \quad (3.1)$$

$$\left\{ \begin{array}{l} \text{for any } x \in \partial K \text{ and } \varepsilon > 0 \text{ there exists } \eta_x > 0 \text{ such that} \\ \text{for all } y, z \in K \cap B(x; \eta_x), \sup_{n \in N_K(y) \cap S^{n-1}} \langle n, z - y \rangle \leq \varepsilon |z - y|. \end{array} \right. \quad (3.2)$$

Remark 3.1. (a) A proximally smooth set, and in particular any convex set, satisfies property (3.2) (cf. [18]). We recall that a closed set $K \subset \mathbb{R}^n$ is called proximally smooth if for some $r > 0$ the distance function to K is continuously differentiable on $\{x \in \mathbb{R}^n : 0 < \text{dist}(x; K) < r\}$.

(b) Assume that $K = \bigcap_{j=1}^m K_j$ is an intersection of a finite number of closed subsets $K_j \subset \mathbb{R}^n$ such that for every j the associated oriented distance function $d_j := d_{K_j} \in C^1$ on a neighborhood \mathcal{V} of ∂K_j with ∇d_j locally Lipschitz on \mathcal{V} . If in addition for all $x \in \partial K$

$$0 \notin \text{co}\{\nabla d_j(x) \mid j \text{ is so that } x \in \partial K_j\},$$

then it is not difficult to show that (3.2) holds true.

(c) Assumption (3.2) implies that K is sleek. Indeed, (3.2) and the characterization of normal cones (cf. [4, Proposition 4.4.1]) yield $N_K(x) \subset T_K(x)^-$ for all $x \in K$ and, therefore, $N_K(x) = T_K(x)^-$. So, thanks to the Bipolar Theorem (see for instance [4, Theorem 2.5.7]),

we obtain $C_K(x) = N_K(x)^- = T_K(x)^{- -} = \overline{co} T_K(x)$. On the other hand, since $C_K(x) \subset T_K(x)$, we deduce that $\overline{co} T_K(x) \subset T_K(x)$ and, therefore $C_K(x) = T_K(x)$.

We first state our main result concerning all extremal pairs. If one is interested by a particular trajectory–control pair (z, \bar{u}) , then it follows from the proofs provided below that both (3.1) and (3.2) may be localized around $z([0, 1])$.

The maximum principle proved in [20], when applied to the present problem, is nondegenerate.

Theorem 3.2. *Suppose that assumptions (3.1) and (3.2) are satisfied. If (z, \bar{u}) satisfies a nondegenerate maximum principle and z is Lipschitz, then this maximum principle is normal.*

To prove the above theorem we need two lemmas in which we impose all the assumptions of Theorem 3.2.

Lemma 3.3. *Suppose that there exists a solution $w \in W^{1,1}$ to the viability problem*

$$\begin{cases} w'(t) \in A(t)w(t) + T_{\overline{co}(f(t, z(t), U(t)))}(z'(t)) & \text{a.e. in } [0, 1], \\ w(0) = 0, \\ w(t) \in \text{Int}(T_K(z(t))) & \text{for all } t \in (0, 1]. \end{cases}$$

Then this nondegenerate maximum principle is normal.

For the basic results on differential inclusions see for instance [3].

The proof of Lemma 3.3 is a slight variation of the one of [14, Lemma 6.1] and for this reason is omitted.

Hence to prove our theorem we have to find a solution to the system of Lemma 3.3.

Lemma 3.4. *Assume that $z((0, 1]) \cap \partial K \neq \emptyset$. Then there exist $\theta > 0$ with $z(\theta) \in \partial K$ and an essentially bounded measurable selection $v(t) \in f(t, z(t), U(t))$ such that the following differential inclusion admits a solution:*

$$\begin{cases} w'(t) \in A(t)w(t) + \mathbb{R}_+(v(t) - z'(t)) & \text{almost everywhere in } [0, \theta], \\ w(0) = 0, \\ w(t) \in \text{Int}(T_K(z(t))) & \text{for all } t \in (0, \theta]. \end{cases} \quad (3.3)$$

In particular $w(\theta) \neq 0$.

Proof. Let M_z be a Lipschitz constant of z . Since K is sleek (cf. Remark 3.1(c)), the set-valued map $K \ni x \rightsquigarrow N_K(x) \cap S^{n-1}$ is upper semicontinuous. Using this fact, the measurable selection theorem (cf. [4, Theorem 8.1.3]) and assumption (3.1)(vi) it is not difficult to show that there exist $\rho > 0$, $\eta > 0$, $M > M_z$ and a measurable selection $v(s) \in f(s, z(s), U(s))$ such that for all $t \in [0, 1] \cap z^{-1}(\partial K)$ and $s \in [t - \eta, t + \eta] \cap [0, 1]$

$$\sup_{n \in N_K(z(t)) \cap S^{n-1}} \langle n, v(s) \rangle \leq -\rho \quad \text{and} \quad \|v\|_\infty \leq M.$$

For any $t_0 \in [0, 1]$, we denote by $Y(\cdot; t_0)$ the fundamental solution to the matrix differential equation

$$\begin{cases} X'(t) = A(t)X(t), \\ X(t_0) = \mathbb{I}. \end{cases} \quad (3.4)$$

Since A is integrable, there exists $0 < \varepsilon_1 < \eta$ (independent from $t_0 \in [0, 1]$) such that whenever $|t - s| \leq \varepsilon_1$, then

$$\|Y(t; t_0)Y^{-1}(s; t_0) - \mathbb{I}\| \leq \frac{\rho}{4(\|v - z'\|_{L^\infty} + 1)}.$$

Let us define $t_1 := \inf z^{-1}(\partial K)$. By (3.2) there exists $0 < \eta_1 < \eta$ such that for all $x, y \in K \cap B(z(t_1); \eta_1)$ if $n \in N_K(y) \cap S^{n-1}$, then

$$\langle n, x - y \rangle \leq \frac{\rho}{4M}|x - y|. \quad (3.5)$$

Let $0 < \varepsilon_0 < \varepsilon_1$ be so that $z([0 \wedge (t_1 - \varepsilon_0), t_1 + \varepsilon_0]) \subset B(z(t_1), \eta_1)$ and $t_0 := 0 \wedge (t_1 - \varepsilon_0)$.

Case 1: $z(0) \notin \partial K$. Consider the system

$$\begin{cases} w'(t) = A(t)w(t) + v(t) - z'(t), \\ w(t_0) = 0. \end{cases} \quad (3.6)$$

Notice that the solution $w \equiv 0$ to $w' = A(s)w$ satisfies $w(t) \in \text{Int}(T_K(z(t))) = \mathbb{R}^n$ on $(0, t_0]$ and for $Z(\cdot) = Y(\cdot; t_0)$ the mapping

$$w(t) = \int_{t_0}^t Z(t)Z^{-1}(s)(v(s) - z'(s))ds$$

is the solution to (3.6) on $[t_0, t_1 + \varepsilon_0]$.

Therefore, for any $n \in N_K(z(t_1)) \cap S^{n-1}$ we get

$$\begin{aligned} \langle n, w(t_1) \rangle &= \int_{t_0}^{t_1} \langle n, Z(t_1)Z^{-1}(s)(v(s) - z'(s)) \rangle ds \\ &\leq \int_{t_0}^{t_1} \langle n, v(s) - z'(s) \rangle ds + \int_{t_0}^{t_1} \|Z(t_1)Z^{-1}(s) - \mathbb{I}\| |v(s) - z'(s)| ds \\ &\leq -\rho(t_1 - t_0) + \langle n, z(t_0) - z(t_1) \rangle + \int_{t_0}^{t_1} \frac{\rho|v(s) - z'(s)|}{4(\|v - z'\|_{L^\infty} + 1)} ds \\ &\leq -\frac{3\rho}{4}(t_1 - t_0) + \frac{\rho}{4M}|z(t_1) - z(t_0)| \leq -\frac{\rho}{2}(t_1 - t_0). \end{aligned} \quad (3.7)$$

Then, we take $\theta = t_1$.

Case 2: $z(0) \in \partial K$ and $(0, \varepsilon_0] \cap z^{-1}(\partial K) = \emptyset$. Then denoting by $t_2 := \inf\{t \in [\varepsilon_0, 1] \cap z^{-1}(\partial K)\}$, we apply the same arguments as in case 1 with t_2 instead of t_1 and we take $\theta = t_2$.

Case 3: $z(0) \in \partial K$ and $(0, \varepsilon_0] \cap z^{-1}(\partial K) \neq \emptyset$. Similarly as for estimate (3.7), for any $t \in [0, \varepsilon_0] \cap z^{-1}(\partial K)$, and thanks to (3.5), for any $n \in N_K(z(t)) \cap S^{n-1}$ we have

$$\begin{aligned} \langle n, w(t) \rangle &= \int_0^t \langle n, Z(s)Z^{-1}(s)(v(s) - z'(s)) \rangle ds \\ &\leq -\rho t + \langle n, z(0) - z(t) \rangle + \int_0^t \frac{\rho|v(s) - z'(s)|}{4(\|v - z'\|_{L^\infty} + 1)} ds \\ &\leq -\frac{3\rho}{4}t + \frac{\rho}{4M}|z(t) - z(0)| \leq -\frac{\rho}{2}t. \end{aligned} \quad (3.8)$$

Setting $\theta = \sup(z^{-1}(\partial K) \cap (0, \varepsilon_0])$ we end our proof. \square

Proof of Theorem 3.2. Thanks to Lemma 3.3 it is sufficient to show that there exist an integrable selection $v(t) \in f(t, z(t), U(t))$ and a solution $w : [0, 1] \rightarrow \mathbb{R}^n$ to the viability problem

$$\begin{cases} w'(t) \in A(t)w(t) + \mathbb{R}_+(v(t) - z'(t)) & \text{a.e. in } [0, 1], \\ w(0) = 0, \\ w(t) \in \text{Int}(T_K(z(t))) & \text{for all } t \in (0, 1]. \end{cases} \quad (3.9)$$

In what follows we construct such a v and w .

If $z((0, 1]) \subset \text{Int}(K)$, then set $v(t) = z'(t)$ a.e. and $w \equiv 0$. We continue under the assumption that $z((0, 1]) \cap K \neq \emptyset$. Let $M, \eta, \rho, \varepsilon_1, v(\cdot), Y(\cdot; \cdot)$ be as in the proof of Lemma 3.4.

Recall that by Lemma 3.4 there exist $\theta > 0$ and a solution w to system (3.9) on $[0, \theta]$ such that $z(\theta) \in \partial K$. Set $w_0 := w(\theta) \neq 0$. For all $s \in z^{-1}(\partial K) \cap [\theta, 1]$ there exists $0 < \eta_s \leq \eta$ such that for any $x, y \in K \cap B(z(s); \eta_s)$ if $n \in N_K(y) \cap S^{n-1}$ then

$$\langle n, x - y \rangle \leq \frac{\rho}{4M}|x - y|. \quad (3.10)$$

So, for each $s \in z^{-1}(\partial K) \cap [\theta, 1]$ we define $\delta_s := \min\{\frac{\eta_s}{M}; \frac{\varepsilon_1}{2}\}$. In such a way we obtain a covering of $z^{-1}(\partial K) \cap [\theta, 1]$

$$\bigcup_{s \in z^{-1}(\partial K) \cap [\theta, 1]} (s - \delta_s, s + \delta_s) \supset z^{-1}(\partial K) \cap [\theta, 1],$$

from which we can extract a finite subcovering $\{(s_i - \delta_i, s_i + \delta_i)\}_{i=1}^N$ with $s_i < s_{i+1}$. For each i define $I_i := [s_i - \delta_i, s_i + \delta_i] \cap [\theta, 1]$.

Step 1. Set

$$\sigma_0 = - \sup_{n \in N_K(z(\theta)) \cap S^{n-1}} \langle n, w_0 \rangle > 0$$

and

$$\sigma := \frac{\sigma_0}{4|w_0|}.$$

Let $Z(\cdot) = Y(\cdot; \theta)$. Since K is sleek (cf. Remark 3.1(c)), $s \rightsquigarrow N_K(z(s)) \cap S^{n-1}$ is upper semicontinuous. Thus, there exists $0 < \alpha < \delta_1$ such that for all $s \in I_1 \cap (\theta, \theta + \alpha)$

$$\begin{cases} \text{(i)} & N_K(z(s)) \cap S^{n-1} \subset N_K(z(\theta)) \cap S^{n-1} + \sigma B, \\ \text{(ii)} & \|Z(s) - \mathbb{I}\| \leq \sigma. \end{cases} \quad (3.11)$$

Define

$$\theta_1 := \sup(z^{-1}(\partial K) \cap I_1), \quad M_1 := \frac{4\|Z\|_\infty|w_0|}{\alpha\rho}.$$

We extend w on $[\theta, \theta_1]$ by the solution to the following system:

$$\begin{cases} w'(t) = A(t)w(t) + M_1(v(t) - z'(t)), \\ w(\theta) =: w_0 \neq 0. \end{cases} \quad (3.12)$$

We claim that $w(t) \in \text{Int}(T_K(z(t)))$ for all $t \in [\theta, \theta_1]$. Indeed, since the solution w to (3.12) is given by the expression

$$w(t) = Z(t)w_0 + M_1 \int_{\theta}^t Z(t)Z^{-1}(s)(v(s) - z'(s)) ds,$$

then for all $t \in z^{-1}(\partial K) \cap I_1$ and $n_t \in N_K(z(t)) \cap S^{n-1}$ we have

$$\langle n_t, w(t) \rangle = \langle n_t, Z(t)w_0 \rangle + M_1 \int_{\theta}^t \langle n_t, Z(t)Z^{-1}(s)(v(s) - z'(s)) \rangle ds.$$

Now, we distinguish two different cases.

Case 1: $t \in z^{-1}(\partial K) \cap I_1$ and $t \geq \theta + \alpha$. Then, we obtain

$$\begin{aligned} \langle n_t, w(t) \rangle &\leq \|Z\|_\infty|w_0| + M_1 \int_{\theta}^t \langle n_t, v(s) - z'(s) \rangle ds \\ &\quad + M_1 \int_{\theta}^t \|Z(t)Z^{-1}(s) - \mathbb{I}\| |v(s) - z'(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq \|Z\|_\infty |w_0| - \rho M_1(t - \theta) + M_1 \langle n_t, z(\theta) - z(t) \rangle + \frac{\rho M_1}{4} \int_\theta^t \frac{|v(s) - z'(s)|}{\|v - z'\|_{L^\infty} + 1} ds \\
&\leq \|Z\|_\infty |w_0| - \frac{1}{2} \rho M_1(t - \theta) \leq -\|Z\|_\infty |w_0|.
\end{aligned} \tag{3.13}$$

Case 2: $t \in z^{-1}(\partial K) \cap (\theta, \theta + \alpha)$. In this case, by property (3.11)(i) for each $n_t \in N_K(z(t)) \cap S^{n-1}$ there exists $n_\theta \in N_K(z(\theta)) \cap S^{n-1}$ such that $|n_\theta - n_t| \leq \sigma$. So, similarly to (3.13) we derive

$$\begin{aligned}
\langle n_t, w(t) \rangle &\leq \langle n_t - n_\theta, w_0 \rangle + \langle n_t, (Z(t) - \mathbb{I})w_0 \rangle + \langle n_\theta, w_0 \rangle \\
&\quad + M_1 \int_\theta^t \langle n_t, v(s) - z'(s) \rangle ds + M_1 \int_\theta^t \|Z(t)Z^{-1}(s) - \mathbb{I}\| |v(s) - z'(s)| ds \\
&\leq 2\sigma |w_0| - \sigma_0 - \frac{1}{2} \rho M_1(t - \theta) \leq -\frac{1}{2} \sigma_0.
\end{aligned} \tag{3.14}$$

Define

$$w_1 := w(\theta_1).$$

If $z^{-1}(\partial K) \cap (\theta_1, 1] = \emptyset$, then it is enough to consider the solution to system

$$\begin{cases} w'(t) = A(t)w(t), \\ w(\theta_1) = w_1 \neq 0 \end{cases} \tag{3.15}$$

on the interval $[\theta_1, 1]$ and to observe that $w(t) \in \text{Int}(T_K(z(t))) = \mathbb{R}^n$ for all $t > \theta_1$. So, we continue our construction assuming that $z^{-1}(\partial K) \cap (\theta_1, 1] \neq \emptyset$.

Step 2. Define

$$t_2 := \inf z^{-1}(\partial K) \cap I_2.$$

We distinguish the following two cases.

Case 1: $\theta_1 < t_2$. Define $Z(\cdot) = Y(\cdot; \theta_1)$ and $\beta := \min\{\delta_2, t_2 - \theta_1\}$.

If $\beta = t_2 - \theta_1$, then we extend w by solving the following system on $[\theta_1, t_2]$:

$$\begin{cases} w'(t) = A(t)w(t) + M_2(v(t) - z'(t)) & \text{a.e. in } [\theta_1, t_2], \\ w(\theta_1) =: w_1 \neq 0, \end{cases} \tag{3.16}$$

where $M_2 := \frac{4\|Z\|_\infty |w_1|}{\rho\beta}$.

Then the solution to (3.16) is given by

$$w(t) = Z(t)w_1 + M_2 \int_{\theta_1}^t Z(t)Z^{-1}(s)(v(s) - z'(s)) ds.$$

Therefore, as in (3.13) for each $n_2 \in N_K(z(t_2)) \cap S^{n-1}$ we get

$$\begin{aligned} \langle n_2, w(t_2) \rangle &= \langle n_2, Z(t_2)w_1 \rangle + M_2 \int_{\theta_1}^{t_2} \langle n_2, Z(t_2)Z^{-1}(s)(v(s) - z'(s)) \rangle ds \\ &\leq \|Z\|_\infty |w_1| - \rho M_2(t_2 - \theta_1) + M_2 \langle n_2, z(\theta_1) - z(t_2) \rangle + \rho M_2 \frac{t_2 - \theta_1}{4} \\ &\leq \|Z\|_\infty |w_1| - \frac{\rho}{2} M_2(t_2 - \theta_1) \leq -\|Z\|_\infty |w_1|. \end{aligned} \quad (3.17)$$

On the other hand, if $\beta < t_2 - \theta_1$, then extend w on $[\theta_1, t_2 - \beta]$ by the solution to (3.15) and define $Z(\cdot) = Y(\cdot; t_2 - \beta)$. Since $z(t) \in \text{Int}(K)$ for all $t \in [\theta_1, t_2 - \beta]$ we get $0 \neq w(t) \in \text{Int}(T_K(z(t))) = \mathbb{R}^n$. Setting $\bar{w}_1 := w(t_2 - \beta)$ and $\bar{M}_2 := \frac{4\|Z\|_\infty|\bar{w}_1|}{\rho\beta}$, consider the solution to system (3.16) with w_1 and M_2 replaced by \bar{w}_1 and \bar{M}_2 , respectively. Exactly as in (3.17) for all $n_2 \in N_K(z(t_2)) \cap S^{n-1}$ we obtain $\langle n_2, w(t_2) \rangle \leq -\|Z\|_\infty |\bar{w}_1|$.

Finally, we apply arguments of Step 1 with t_2 instead of θ in order to get a solution to (3.9) with $[0, 1]$ replaced by $[0, \sup(z^{-1}(\partial K) \cap I_2)]$.

Case 2: $\theta_1 \geq t_2$. We extend w on $[\theta_1, \sup(z^{-1}(\partial K) \cap I_2)]$ as in Step 1 replacing θ by θ_1 .

Step 3. We complete the construction of the solution to problem (3.9) by the induction argument. Indeed, assume that for some $1 < k < N$ we have already defined $w : [0, \sup(z^{-1}(\partial K) \cap I_k)] \rightarrow \mathbb{R}^n$ satisfying (3.9). Define $\theta_k := \sup z^{-1}(\partial K) \cap I_k$ and $w_k := w(\theta_k)$. Then, applying the same arguments as in Step 2 we extend w on $[\theta_k, \sup(z^{-1}(\partial K) \cap I_{k+1})]$.

In this way we obtain a solution to (3.9) on $[0, \theta_N]$, where $\theta_N = \sup(z^{-1}(\partial K) \cap I_N)$. Set $w_N = w(\theta_N)$ and extend w on $[\theta_N, 1]$ by the solution to $w' = A(s)w$, $w(\theta_N) = w_N$. The proof is complete. \square

Remark 3.5. Using similar arguments, it can be shown that if $z \in C^1$, then it may be assumed that K is only sleek (instead of (3.2)) to deduce normality of the maximum principle.

The next result provides the positive answer to Ioffe's question.

Corollary 3.6. *Suppose that assumptions (3.1) and (3.2) are satisfied. Then every extremal trajectory/control pair (z, \bar{u}) of system (1.1) with Lipschitz z satisfying $z(0) \in \text{Int}(K)$ is normal.*

Proof. Let λ, p, ψ be as in Definition 2.1. Assume for a moment that $\lambda = 0$. Let w be as constructed in the proof of Theorem 3.2. Then for some $\zeta(t) \in T_{\overline{\text{co}}(f(t, z(t), U(t)))}(z'(t))$ we have $w'(t) = A(t)w(t) + \zeta(t)$ a.e. Using the maximum principle and integrating by parts we obtain

$$\begin{aligned} 0 &= \langle p(1) + \psi(1), w(1) \rangle = \int_0^1 (\langle p'(t), w(t) \rangle + \langle p(t), w'(t) \rangle) dt + \langle \psi(1), w(1) \rangle \\ &= \int_0^1 (\langle -A(t)^*(p(t) + \psi(t)), w(t) \rangle + \langle p(t), A(t)w(t) + \zeta(t) \rangle) dt + \langle \psi(1), w(1) \rangle \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 (\langle \psi(t), -w'(t) + \zeta(t) \rangle + \langle p(t), \zeta(t) \rangle) dt + \langle \psi(1), w(1) \rangle \\
&= \int_0^1 w(t) d\psi(t) + \int_0^1 \langle p(t) + \psi(t), \zeta(t) \rangle dt \leq \int_{[0,1]} \langle w(t), v(t) \rangle d\mu(t).
\end{aligned}$$

Using that $w(t) \in \text{Int}(T_K(z(t)))$ for all $t \in (0, 1]$ we deduce that $v(t) = 0$ μ -a.e. in $(0, 1]$. Finally, from $z(0) \in \text{Int}(K)$ we obtain $\psi = 0$. This and the adjoint equation yield $p = 0$, contradicting the condition $(\lambda, p, \psi) \neq 0$. \square

Corollary 3.7. *Suppose that assumptions (3.1) and (3.2) are satisfied. If a trajectory/control pair (z, \bar{u}) of system (1.1) with Lipschitz z satisfies a maximum principle with ψ not equal to a constant on $(0, 1]$, then this maximum principle is normal.*

Proof. Exactly as in the proof of Corollary 3.6 we show that if $\lambda = 0$, then $v(t) = 0$ μ -a.e. in $(0, 1]$, which implies that ψ is constant on $(0, 1]$, contradicting our assumption. \square

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